Young Women in Algebraic Geometry

Around Hodge, Tate and Mumford-Tate conjectures on abelian varieties

Victoria Cantoral-Farfán

Advisor : Prof. Marc Hindry

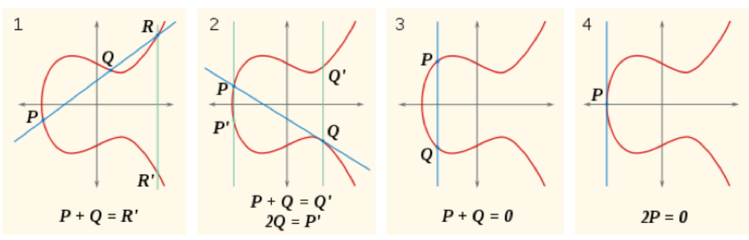
IMJ-PRG Institut de Mathématiques de Jussieu-Paris Rive Gauche

CONACYT

Preliminaries

Definition. An **abelian variety** is a projective algebraic variety that is also an algebraic group with a group law which is commutative.

Example. *An elliptic curve is an abelian variety of dimension* 1 *with a group law which is explained here:*



Let define $V = H_1(X_{\mathbb{C}}, \mathbb{Q})$ and $V_l := T_l(X) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ where the l-adic Tate module is defined as follows $T_l(X) := \varprojlim X[l^n]$ and where $X[l^n]$ is the Kernel of the multiplication by l^n. By the comparison theorem we have $V_l \simeq V \otimes_{\mathbb{Q}} \mathbb{Q}_l$.

We recall that the cohomology group $H^{2p}(X_{\mathbb{C}}, \mathbb{Q})$ of $X_{\mathbb{C}}$ is naturally endowed with a Hodge structure of weight $2p$.

$$H^{2p}(X_{\mathbb{C}}, \mathbb{Q}) \otimes \mathbb{C} = H^{2p}(X_{\mathbb{C}}, \mathbb{C}) = \bigoplus_{r+s=2p} H^{r,s}.$$

Because $X_{\mathbb{C}}$ is an a.v. we have that $H^{2p}(X_{\mathbb{C}}, \mathbb{C}) = \bigwedge^{2p} H^1(X_{\mathbb{C}}, \mathbb{C})$. Moreover $H^1(X_{\mathbb{C}}, \mathbb{C}) = T_0(X_{\mathbb{C}}) \oplus T_0(X_{\mathbb{C}})^{\vee}$ and therefore we have

$$H^{2p}(X_{\mathbb{C}}, \mathbb{C}) = \bigwedge^{2p}(T_0(X) \oplus T_0(X)^{\vee}) = \bigoplus_{r+s=2p} (\bigwedge^{r} T_0(X) \wedge \bigwedge^{s} T_0(X)^{\vee}).$$

Definition. The **group of Hodge classes of codimension** p is defined by

$$B^p(X) := H^{2p}(X_{\mathbb{C}}, \mathbb{Q}) \cap H^{p,p} \subset H^{2p}(X_{\mathbb{C}}, \mathbb{C}).$$

We define $\mathbb{G}_{m,\mathbb{Q}} \subset GL(V)$ as the group of homotheties and $\mathbb{S}$ be the restriction of scalars of $\mathbb{G}_{m,\mathbb{C}}$ from $\mathbb{C}$ to $\mathbb{R}$. Thanks to Hodge decomposition we have $V_{\mathbb{C}} = H_1(X, \mathbb{C}) = V^{-1,0} \oplus V^{0,-1}$ and we can therefore consider the morphism:

$$h : \mathbb{S} \to GL(V)_{\mathbb{R}}; \quad \forall z \in \mathbb{S} \quad h(z)(v^{-1,0}) = z \times v^{-1,0} \quad \& \quad h(z)(v^{0,-1}) = \bar{z} \times v^{0,-1}.$$

Definition. The **Hodge group** $\mathrm{Hg(X)}$ of X is the smallest algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$ such that $h_{|U^1}$ factors through $\mathrm{Hg(X)} \otimes \mathbb{R}$ (where $U^1 \subset \mathbb{S}$ and $U^1(\mathbb{R}) = \{z \in \mathbb{C}^*, z\bar{z} = 1\}$).

Let $G_k = Gal(\bar{k}/k)$ be the absolute Galois group and we define the following l-adic representation:

$$\rho_l : G_k \to Aut_{\mathbb{Q}_l}(V_l)$$

We define the algebraic group G_l as the Zariski closure of the image of ρ_l, i.e. $G_l = \overline{\rho_l(G_k)}^{Zar}$, which is called the algebraic monodromy group at l.

Definition. The **Galois monodromy group** H_l is defined as follows:

$$H_l = (G_l \cap SL(V_l))^{\circ}.$$

Example. *If* X_k *is an elliptic curve then* $\mathrm{Hg(X)}$ *is equal to* $SL_2(V)$ *or to a torus and* H_l *is equal to* $SL_2(V_l)$ *or to a torus.*

Introduction

The aim of this poster is to present Hodge, Tate and Mumford-Tate conjectures focusing on abelian varieties and to describe the links between them. Furthermore we are going to illustrate some known results. Hodge conjecture was introduced in 1950, its main goal is to establish a bridge between Algebraic Geometry and Differential Geometry. Hodge was inspired by Lefschetz's theorems. This conjecture is stated for algebraic varieties which are projective and smooth. Nevertheless it is easier to describe and even to establish some results about this conjecture for complex abelian varieties. One of the reasons is the simple decomposition of the singular cohomology of an abelian variety and its Hodge decomposition. Moreover we can wonder if an equivalent of this conjecture exists in Arithmetic Geometry. The answer is the Tate conjecture, stated in 1963. As for the previous conjecture, we will focus our attention on abelian varieties this time over a number field k. Instead of the singular cohomology we use the étale l-adic cohomology for abelian varieties.

Motivations

Let X_k be an abelian variety over a number field k of dimension g and let us fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ such that $X_{\mathbb{C}} = X_k \underset{k,\sigma}{\times} \mathbb{C}$. Let $p \in [\![0, g]\!]$ and l be a prime number.

Hodge conjecture (H)
for a.v. over $\mathbb{C}$
'50

Conjecture. Each Hodge classes are $\mathbb{Q}$-linear combinations of algebraic classes.

Hodge classes $= H^{\bullet}(X_{\mathbb{C}}, \mathbb{Q})^{\mathrm{Hg(X)}}$.

Tate conjecture (T)
for a.v. over k
'63

Conjecture. Each Tate classes are $\mathbb{Q}_l$-linear combinations of algebraic classes.

Tate classes $:= H^{\bullet}_{et}(X_k, \mathbb{Q}_l)^{H_l}$.

Mumford-Tate conjecture (MT)
'66

Conjecture. $\mathrm{Hg(X)} \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq \mathrm{H}_l \quad \forall \ell$.

One can remark that the first equality of Hodge classes is actually a theorem whereas the second equality is, as a matter of fact, the definition of Tate classes.

Definition. The **group of algebraic classes of codimension** p is defined as follows:

$$C^p(X) = cl(Z^p(X)_{\mathbb{Q}}) \subset H^{2p}(X, \mathbb{Q}),$$

where $Z^p(X)_{\mathbb{Q}}$ is the group generated by subvarieties of X of codimension p and $cl : Z^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q})$ is the cycle map.

Example. *Divisors are algebraic classes of codimension* 1. *Moreover due to Lefschetz and Faltings theorems Hodge and Tate conjecture holds for divisoirs.*

Links and results for the three conjectures

Piatetskii-Shapiro proved the following equivalence, the first implication is easier to explain, it follows directly by definitions.

$$(H) + (MT) \Leftrightarrow (T)$$

For instance, in these two examples Tate conjecture (T) follows from by Hodge (H) and Mumford-Tate (MT) conjectures.

Example.

X an a.v. of prime dimension
Tankeev & Ribet '83
Chi '91

X an a.v. of type I or II on the Albert's classification with odd relative dimension.
Murty & Hazama '84
Banaszak, Gajda & Krason '03

Some others results

- First results about those three conjectures:

X an a.v. such that $dim\, X \leq 3$
Lefschetz '24
Faltings '83

X an elliptic curve
Serre '72

- Some results on (H) have been prooved conecerning the product of elliptic curves not isogenous and abelian fourfold (except some special cases).

  Example. *Imai '76 and Moonen & Zarhin '99.*

- Some results on (MT) with conditions on the type of the endomorphism ring End(X) and the dimension of X.

  Example. *Serre '84, Chi '92, Pink '98 and Banaszak, Gajda & Krason '03.*

- Others results about (MT) with condition on the toric dimension of a semi stable fiber of the Néron model of X.

  Example. *Hall '08 and Hindry & Ratazzi '15.*

- Finally, some results of (T) can be deduce thanks to this previous equivalence.

References

1. B. Gordon, *A survey of the Hodge conjecture for Abelian Varieties*, Appendix B, in: *A Survey of the Hodge Conjecture*, J. Lewis, CRM monograph series, Second edition.
2. J. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, (1965), 93–110.
3. D. Mumford, *Families of abelian varieties*, Proc.of Symposia in Pure Math. A.M.S., **IX** (1966), 347–351.

2015 Institut de Mathématiques de Jussieu - Paris Rive Gauche (IMJ-PRG), Paris, France
victoria.cantoral-farfan@imj-prg.fr