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Fields of definition of elliptic fibrations on covers of certain extremal rational elliptic surfaces

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0 Joint work with

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0 Outline

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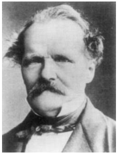
1 Outline

1. General introduction & motivations
   - K3 surfaces
   - Elliptic fibrations
2. Preliminaries, setting & goals
3. Results & examples

1 K3 surfaces

Erich Kähler

Kunihiko Kodaira



Ernst Kummer

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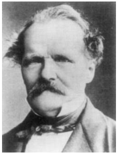

Ernst Kummer

Definition

An algebraic K3 surface X is a smooth, projective, 2-dimensional variety defined over a field k such that :

- $\omega_X \simeq \mathcal{O}_X$,
- $H^1(X, \mathcal{O}_X) = 0$.

1 K3 surfaces

Examples

1. A smooth quartic surface in $\mathbb{P}^3_k$.
2. A Kummer surface.

1 Elliptic fibrations

Definition

An elliptic fibration of a surface S is a surjective morphism

$$\mathcal{E} : S \to C$$

where C is a smooth curve defined over the field k, such that:

1. almost all the fibers are smooth genus 1 curves,
2. at least one singular fiber,
3. at least one section (the zero section).

1 Elliptic fibrations

1 Elliptic fibrations

1. Denote by E the general fiber of $\mathcal{E}$; which is an elliptic curve defined over the function field $k(C)$.
2. Denote by $\mathrm{MW}(\mathcal{E})$ the Mordell–Weil group of $\mathcal{E}$:

$$\mathrm{MW}(\mathcal{E}) = E(k(C)) = \{\text{sections of } \mathcal{E} : S \to C\}.$$

2 Outline

2 Rational elliptic surfaces

Definition

A rational elliptic surface R is a smooth rational surface endowed with an elliptic fibration $\mathcal{E}_R : R \to \mathbb{P}^1$.

Example

A pencil of cubics in $\mathbb{P}^2_{\mathbb{Q}}$

2 Extremal rational elliptic surfaces

Definition

An extremal rational elliptic surface R is such that

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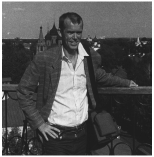

Ulf Persson

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Theorem (Miranda & Persson 1996)

There exist only 16 fiber configurations of extremal rational elliptic surfaces.

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- the two branched fibers $\mathcal{E}_R^{-1}(P_1)$ and $\mathcal{E}_R^{-1}(P_2)$ are smooth and reduced fibers, $G_{\bar{k}}$-conjugate.
- $X \simeq R \times_d \mathbb{P}^1$ is a K3 surface def. / k;
- the double cover d induces the elliptic fibration $\mathcal{E}_X$ & the zero section $\mathcal{O}_X$ both def. / k.

2 Remarks and notations

- $R \times_d \mathbb{P}^1$ is endowed with an involution which is the cover involution of $R \times_d \mathbb{P}^1 \to R$ induced by d.
- This involution can be extended to an involution of $X \simeq R \times_d \mathbb{P}^1$ denoted $\tau \in \mathrm{Aut}(X)$.
- By construction, τ is a non-symplectic involution on X (i.e. does not preserves the symplectic form defined on X).
- Denote by k_τ/k the quadratic extension of k such that Gal$(k_\tau/k) =< \tau >$.

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2. For each elliptic fibration on X
   1. determine its field of definition i.e. the field over which the class of a fiber & a section are defined;
   2. give an upper bound for the degree of the field over which the Mordell–Weil group of the fibration admits a set of generators.

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We prove that a genus 1 fibration on X admits a section over a field which depends on the **action** of the cover involution τ on the fibers of the genus 1 fibration.

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2. For each elliptic fibration on X
   1. determine the **type** of the fibration w.r.t. the cover involution τ and **hence** the field of definition of the fibration;
   2. first determine the Mordell–Weil group of the fibration and then give an upper bound for the degree of the field over which the Mordell–Weil group admits a set of generators.

2 Some extra definition

Definition

Let η be an elliptic fibration on X then it is

- of type 1 with respect to τ, if τ preserves all the fibers of η;
- of type 2 with respect to τ, if τ does not preserve all the fibers of η, but maps a fiber of η to another one. In this case τ is induced by an involution of the basis of $\eta : X \to \mathbb{P}^1$. It fixes exactly two fibers and τ^{*},[1] preserves the class of a fiber of η;
- of type 3, if τ maps fibers of η to fibers of another elliptic fibration. In this case τ^* does not preserve the class of the generic fiber of η.

[1]We denote by τ^* the involution induced by τ on $\mathsf{NS}(X)$

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Lemma (C.-F.,Garbagnati, Salgado, Winter, Trbović)

Let R be an extremal rational elliptic surface defined over k. Assume that all reducible fibers of the elliptic fibration are distinct. Then the Néron–Severi group $\mathrm{NS}(R)$ admits generators defined over a field extension of k of degree at most 2.

3 Example

- R_9 an extremal rational elliptic surface def. $/\ k$ with reducible fiber of type I_9;
- X_9 a K3 surface, defined over k, obtained by a double cover of R_9 branched in two smooth $G_{\bar{k}}$-conjugate fibers;
- $\mathcal{E}_{R_9}$ the elliptic fibration def. $/\ k$ & $\mathcal{O}_{R_9}$ the zero section def. $/\ k$.

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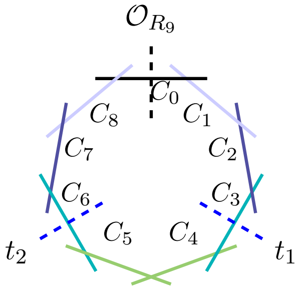
- $\mathcal{E}_{R_9}$ the elliptic fibration def. $/\ k$ & $\mathcal{O}_{R_9}$ the zero section def. $/\ k$.

- The singular fibers of R_9 are $I_9 + 3I_1$.
- The Mordell-Weil group is $\mathbb{Z}/3\mathbb{Z} = \{\mathcal{O}_{R_9}, t_1, t_2\}$.

3 Example



The fiber I_9

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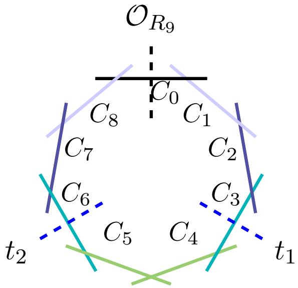

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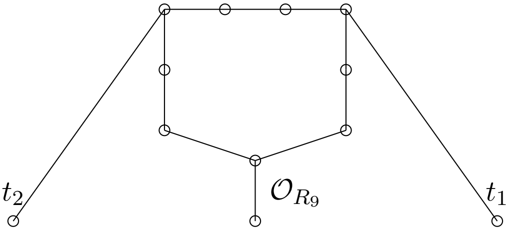

Dual graph $\tilde{A}_8$

3 Idea of the proof

1. Shioda–Tate formula asserts that:

$$\mathrm{NS}(R) \simeq \langle \mathcal{O}, F \rangle \oplus \mathrm{MW}(\mathcal{E}_R) \oplus \sum_{\substack{v \in \text{reducible fibers} \\ i \in S_v}} \Theta_{v,i},$$

where $\mathrm{MW}(\mathcal{E}_R)$ is a finite group, and $\Theta_{v,i}$ are the components of the reducible fiber $\mathcal{E}_R^{-1}(v)$ with n_v its number of components and $S_v = \{0, \cdots, n_v - 1\}$.

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$$\begin{aligned} \mathrm{NS}(R_9) &\simeq \langle \mathcal{O}_{R_9}, F\rangle \oplus \mathrm{MW}(\mathcal{E}_{R_{R_9}}) \oplus \sum_{\substack{v\in \text{reducible fibers} \\ i\in S_v}} \Theta_{v,i} \\ &\simeq \langle \mathcal{O}_{R_9}, F, t_1, t_2, \Theta_0, \Theta_1, \cdots, \Theta_8\rangle. \end{aligned}$$

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$$\simeq \langle \mathcal{O}_{R_9}, F, t_1, t_2, \Theta_0, \Theta_1, \cdots, \Theta_8 \rangle.$$

2. The absolute Galois group $G_{\bar{k}}$ acts on NS(R_9) preserving the intersection pairing.

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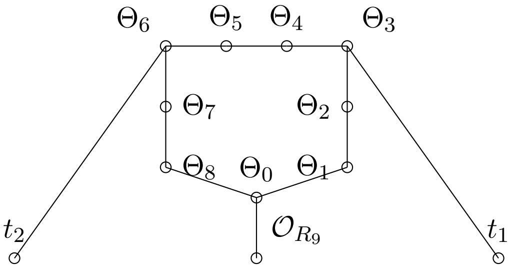
Denote by k_R/k the quad. extension where the fiber components are defined.

3 Come back to the example



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- The point of intersection is def. $/\ k_R$ & the section C has a k_R-point,
  - hence C is def. $/\ k_R$.

3 Results

Lemma (C.-F.,Garbagnati, Salgado, Winter, Trbović)

Let R be an extremal rational elliptic surface defined over k. Assume that all reducible fibers of the elliptic fibration are distinct. Then the Néron-Severi group $\mathrm{NS}(R)$ admits generators defined over a field extension of k of degree at most 2.

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- Indeed, extremal RES with repeated reducible fibers have their Néron-Severi group defined, in general, over extensions of larger degree.

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Let R be an extremal rational elliptic surface defined over k. Assume that all reducible fibers of the elliptic fibration are distinct. Then the Néron-Severi group $\mathrm{NS}(R)$ admits generators defined over a field extension of k of degree at most 2.

- This Lemma is excluding 5 out of 16 configuration of reducible fibers on extremal RES:
  - $(2I_0^*)$, $(2I_5, 2I_1)$, $(2I_4, 2I_2)$, $(I_2^*, 2I_2)$ & $(4I_3)$.
- Indeed, extremal RES with repeated reducible fibers have their Néron-Severi group defined, in general, over extensions of larger degree.

3 Notations

- Let use denote by k_R the quadratic extension of k over which the Néron-Severi group $\mathrm{NS}(R)$ admits a set of generators given by fiber components and sections of the elliptic fibration on R.
- Denote by G_R the Galois group $\mathsf{Gal}(k_R/k)$.
- Let $k_{R,\tau}$ be the compositum of the fields k_R and k_τ.

3 Upshot

Prove that a genus 1 fibration on X admits a section over a field which depends on the **action** of the cover involution τ on the fibers of the genus 1 fibration.

3 Results

Theorem (C.-F.,Garbagnati, Salgado, Winter, Trbović)

Let R be an extremal rational elliptic surface defined over k with distinct reducible fibers. Let X be a K3 surface obtained as a double cover of R branched on two smooth fibers conjugate under $G_{\bar{k}}$, τ the cover involution and η a genus 1 fibration on X. Then the following hold.

i) *If η is of type 1 w.r.t. τ then η is defined over k_R and admits a section over $k_{R,\tau}$.*

ii) *If η is of type 2 then it is defined and admits a section over k.*

iii) *If η is of type 3 then it is defined and admits a section over $k_{R,\tau}$.*

3 Proof - Type 2 w.r.t. τ

Let η be a genus 1 fibration on X.

- Assume that η is of type 2 w.r.t τ.

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Let η be a genus 1 fibration on X.

- Assume that η is of type 2 w.r.t τ.
  - τ does not preserve all the fibers of η, but maps a fiber of η to another one. In this case τ is induced by an involution of the basis of $\eta : X \to \mathbb{P}^1$. It fixes exactly two fibers and τ^* preserves the class of a fiber of η.

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  - Hence, each fiber is the pull-back of a conic C in R (i.e that is a rational curve such that $C.(-K_R) = 2$).

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- The fibers of η are **not** fixed by τ,
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  - Moreover, all negative curves in R are defined over k_R. Hence, S is defined over $k_{R,\tau}$.

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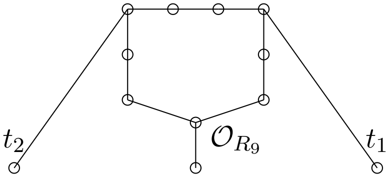

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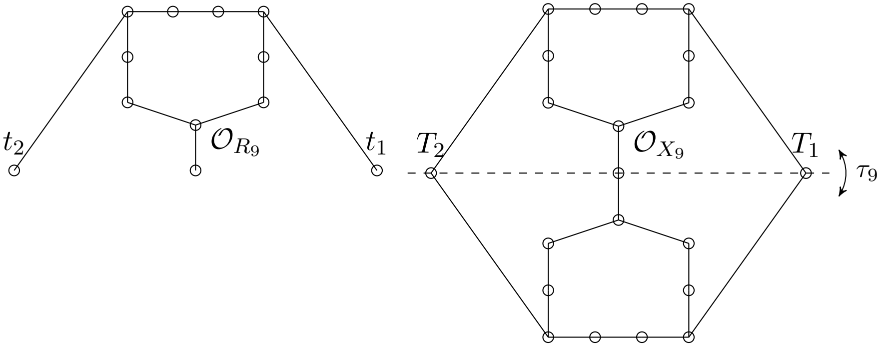

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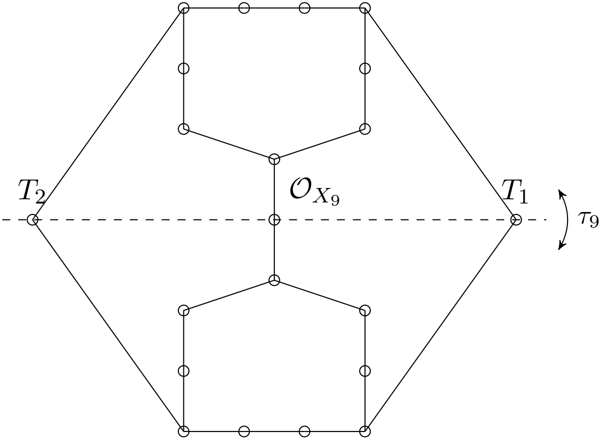

Dank u zeer !