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Fields of definition of elliptic fibrations on covers of certain extremal rational elliptic surfaces

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0 Joint work with

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0 Outline

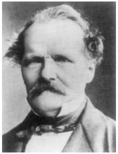
1 Outline

1. General introduction & motivations
   - K3 surfaces
   - Elliptic fibrations
2. Preliminaries, setting & goals
3. Results & examples

1 K3 surfaces

Erich Kähler

Kunihiko Kodaira



Ernst Kummer

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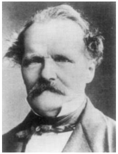

Ernst Kummer

Definition

An algebraic K3 surface X is a smooth, projective, 2-dimensional variety defined over a field k such that :

- $\omega_X \simeq \mathcal{O}_X$,
- $H^1(X, \mathcal{O}_X) = 0$.

1 K3 surfaces

Examples

1. A smooth quartic surface in $\mathbb{P}^3_k$.
2. A Kummer surface.

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     Denote by $\tilde{A}$ the blow up of A along those 16 points and consider the lift $\tilde{\iota} : \tilde{A} \to \tilde{A}$.

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     Denote by $\tilde{A}$ the blow up of A along those 16 points and consider the lift $\tilde{\iota} : \tilde{A} \to \tilde{A}$.
     The Kummer surface associated to A is the K3 surface $X := \tilde{A}/\tilde{\iota}$.

1 Elliptic fibrations

Definition

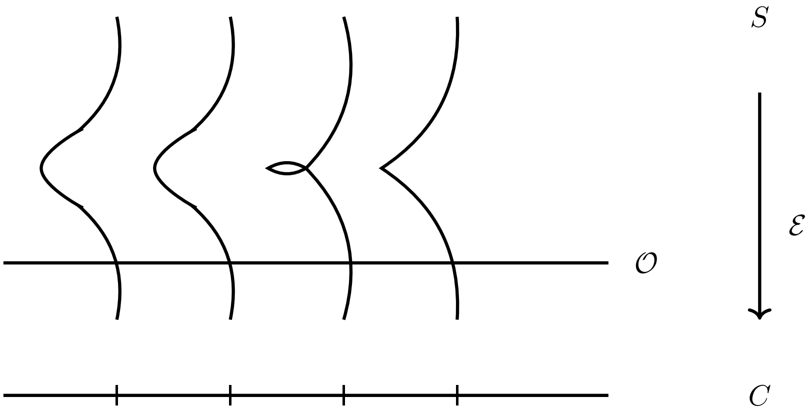
An elliptic fibration of a surface S is a surjective morphism

$$\mathcal{E} : S \to C$$

where C is a smooth curve defined over the field k, such that:

1. almost all the fibers are smooth genus 1 curves,
2. at least one singular fiber,
3. at least one section (the zero section).

1 Elliptic fibrations



1 Elliptic fibrations

1. Denote by E the general fiber of $\mathcal{E}$; which is an elliptic curve defined over the function field $k(C)$.
2. Denote by $\mathrm{MW}(\mathcal{E})$ the Mordell–Weil group of $\mathcal{E}$:

$$\mathrm{MW}(\mathcal{E}) = E(k(C)) = \{\text{sections of } \mathcal{E} : S \to C\}.$$

2 Outline

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Definition

A rational elliptic surface R is a smooth rational surface endowed with an elliptic fibration $\mathcal{E}_R : R \to \mathbb{P}^1$.

Example

A pencil of cubics in $\mathbb{P}^2_{\mathbb{Q}}$

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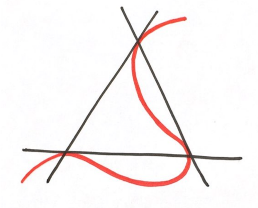

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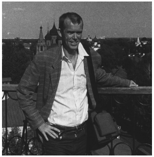

Ulf Persson

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$$\begin{tikzcd} X \simeq R \times_{\mathbb{P}^1} \mathbb{P}^1 \arrow[r] \arrow[d, "\mathcal{E}_X"'] & R \arrow[d, "\mathcal{E}_R"] \\ \mathbb{P}^1 \arrow[r, "d"] & \mathbb{P}^1 \end{tikzcd}$$

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- $\Rightarrow \mathcal{E}_X \,\&\, \mathcal{O}_X$ are def. $/k$.

2 Remarks and notations

- $R \times_{\mathbb{P}^1} \mathbb{P}^1$ is endowed with an involution which is the cover involution of $R \times_{\mathbb{P}^1} \mathbb{P}^1 \to R$ induced by d.
- This involution can be extended to an involution of $X \simeq R \times_{\mathbb{P}^1} \mathbb{P}^1$ denoted $\tau \in \mathrm{Aut}(X)$.
- By construction, τ is a non-symplectic involution on X (i.e. does not preserves the symplectic form defined on X).
- Denote by k_τ/k the quadratic extension of k such that Gal$(k_\tau/k) = < \tau >$.

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We prove that a genus 1 fibration on X admits a section over a field which depends on the **action** of the cover involution τ on the fibers of the genus 1 fibration.

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   2. first determine the Mordell–Weil group of the fibration and then give an upper bound for the degree of the field over which the Mordell–Weil group admits a set of generators.

2 Some extra definition

Definition

Let η be an elliptic fibration on X then it is

- of type 1 with respect to τ, if τ preserves all the fibers of η;
- of type 2 with respect to τ, if τ does not preserve all the fibers of η, but maps a fiber of η to another one. In this case τ is induced by an involution of the basis of $\eta : X \to \mathbb{P}^1$. It fixes exactly two fibers and $\tau^{*,1}$ preserves the class of a fiber of η;
- of type 3, if τ maps fibers of η to fibers of another elliptic fibration. In this case τ^* does not preserve the class of the generic fiber of η.

[1] We denote by τ^* the involution induced by τ on $\mathsf{NS}(X)$

3 Outline

3 Results

Lemma (C.-F.,Garbagnati, Salgado, Winter, Trbović)

Let R *be an extremal rational elliptic surface defined over* k*. Assume that all reducible fibers of the elliptic fibration are distinct. Then the Néron–Severi group* $\mathrm{NS}(R)$ *admits generators defined over a field extension of* k *of degree at most 2.*

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1. Shioda–Tate formula asserts that:

$$\mathrm{NS}(R)/T \simeq \mathsf{MW}(\mathcal{E}_R) \quad \text{with} \quad T = \langle \mathcal{O}, F \rangle \oplus \sum_{\substack{v \in \mathsf{Red} \\ i \in S_v}} \Theta_{v,i},$$

$\Theta_{v,i}$ are the fiber components of the reducible fiber $F_v := \mathcal{E}_R^{-1}(v)$;
Red $:= \{v \in \mathbb{P}^1;\ F_v \text{ reducible}\}$;
n_v the number of fiber components of F_v and
$S_v = \{0, \cdots, n_v - 1\}$.

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2. The absolute Galois group $G_{\bar{k}}$ acts on $\mathrm{NS}(R)$ preserving the intersection pairing.

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    - If F_v has exactly 2 fiber components $\Theta_{v,0}$ and $\Theta_{v,1}$ then they are def. $/k$.

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- Let us denote by $k_{R,v}$ the quadratic extension $/k$ over which the fiber components $\Theta_{v,i}$ of F_v are defined.

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- Let $S \in \mathsf{MW}(\mathcal{E}_R)$, we know that S is a (-1)-curve in $\mathsf{NS}(R)$ and that $\mathsf{MW}(\mathcal{E}_R)$ is globally def. $/k$.

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MW($\mathcal{E}_R$)

- Let $S \in$ MW($\mathcal{E}_R$), we know that S is a (-1)-curve in NS(R) and that MW($\mathcal{E}_R$) is globally def. $/k$.
- Let F_v be a reducible fiber with at least 3 fiber components.
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$\mathrm{MW}(\mathcal{E}_R)$

- Let $S \in \mathrm{MW}(\mathcal{E}_R)$, we know that S is a (-1)-curve in $\mathrm{NS}(R)$ and that $\mathrm{MW}(\mathcal{E}_R)$ is globally def. $/k$.
- Let F_v be a reducible fiber with at least 3 fiber components.
  - S intersects a unique fiber component $\Theta_{v,i}$ of F_v in the point $P := S \cap \Theta_{v,i}$.
- Hence, S is defined either over k or over $k_{R,v}$.

3 Results

Lemma (C.-F.,Garbagnati, Salgado, Winter, Trbović)

Let R *be an extremal rational elliptic surface defined over* k. *Assume that all reducible fibers of the elliptic fibration are distinct. Then the Néron-Severi group* $\mathrm{NS}(R)$ *admits generators defined over a field extension of* k *of degree at most 2.*

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3 Example

II, II^*	III, III^*	IV, IV^*	$2I_0^*$
$II^*, 2I_1$	III^*, I_2, I_1	IV^*, I_3, I_1	$I_4^*, 2I_1$
I_1^*, I_4, I_1	$I_2^*, 2I_2$	$I_9, 3I_1$	$I_8, I_2, 2I_1$
$2I_5, 2I_1$	I_4, I_3, I_2, I_1	$2I_4, 2I_2$	$4I_3$

Table: List of the 16 configurations of singular fibers

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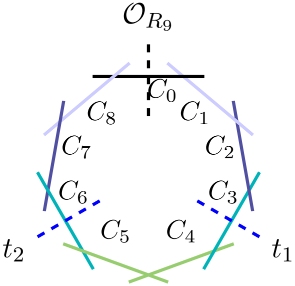
- R_9 an extremal rational elliptic surface def. $/\ k$ with singular fibers $I_9 + 3I_1$;
- $\mathcal{E}_{R_9}$ the elliptic fibration def. $/\ k$ & $\mathcal{O}_{R_9}$ the zero section def. $/\ k$;
- The Mordell-Weil group is $\mathrm{MW}(\mathcal{E}_{R_9}) = \mathbb{Z}/3\mathbb{Z} = \{\mathcal{O}_{R_9}, t_1, t_2\}$.

3 Example



The fiber I_9

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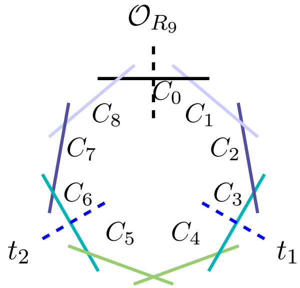

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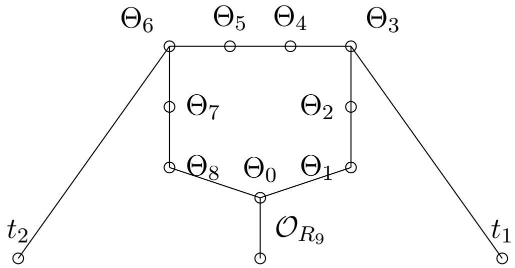

Dual graph $\tilde{A}_8$

3 Notations

- Let use denote by k_R the quadratic extension of k over which the Néron-Severi group $\mathrm{NS}(R)$ admits a set of generators given by fiber components and sections of the elliptic fibration on R.
- Denote by G_R the Galois group $\mathsf{Gal}(k_R/k)$.
- Let $k_{R,\tau}$ be the compositum of the fields k_R and k_τ.

3 Upshot

Prove that a genus 1 fibration on X admits a section over a field which depends on the **action** of the cover involution τ on the fibers of the genus 1 fibration.

3 Results

Theorem (C.-F.,Garbagnati, Salgado, Winter, Trbović)

Let R be an extremal rational elliptic surface defined over k with distinct reducible fibers. Let X be a K3 surface obtained as a double cover of R branched on two smooth fibers conjugate under $G_{\bar{k}}$, τ the cover involution and η a genus 1 fibration on X. Then the following hold.

i) *If η is of <u>type 1</u> w.r.t. τ then η is defined over k_R and admits a section over $k_{R,\tau}$.*

ii) *If η is of <u>type 2</u> then it is defined and admits a section over k.*

iii) *If η is of <u>type 3</u> then it is defined and admits a section over $k_{R,\tau}$.*

3 Proof - Type 2 w.r.t. τ

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Let η be a genus 1 fibration on X.

- Assume that η is of type 2 w.r.t τ.
  - τ does not preserve all the fibers of η, but maps a fiber of η to another one. In this case τ is induced by an involution of the basis of $\eta : X \to \mathbb{P}^1$. It fixes exactly two fibers and τ^* preserves the class of a fiber of η.

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  - Hence, each fiber is the pull-back of a conic C in R (i.e that is a rational curve such that $C.(-K_R) = 2$).

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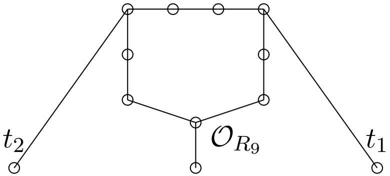

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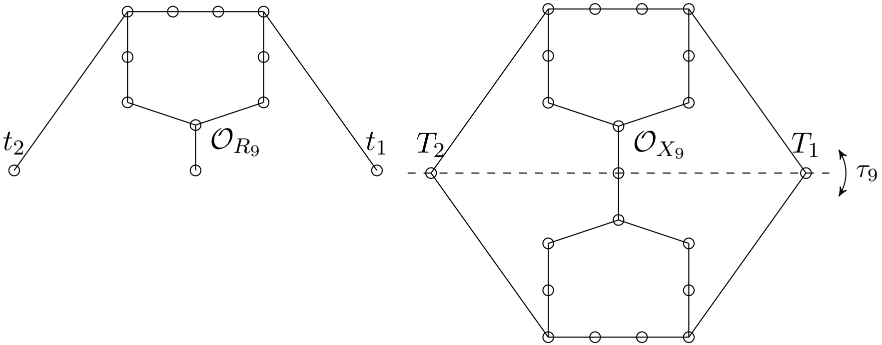

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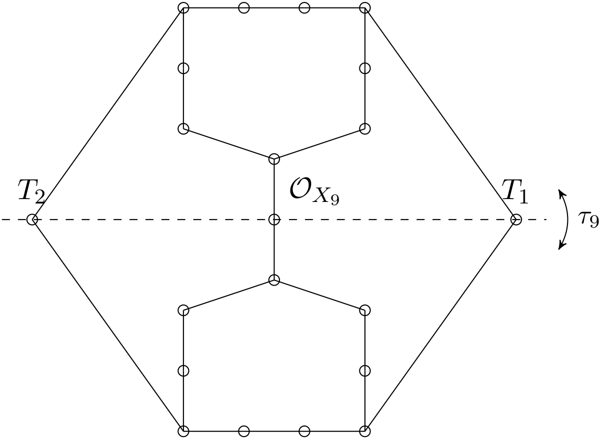

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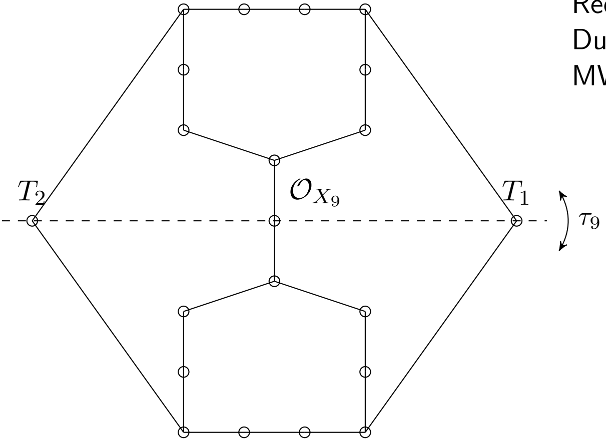

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Reducible fiber I_{16};
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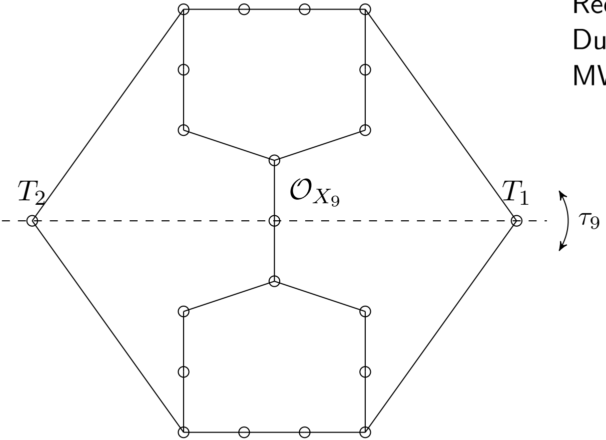

Type 2
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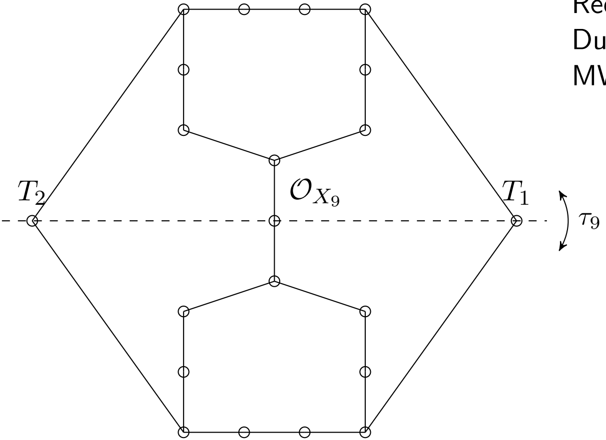

Type 3

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Vielen Dank !